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# Static charged dust distributions with plane symmetry

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**Abstract.** A static line element for which the diagonalized metric tensor components are functions of only one coordinate is considered. It is found that for charged dust distributions the solutions cannot be fitted with an external matter-free space. Further, no regular solution can be obtained for distributions extending over all space. The result is an extension of Som's result for the case of cylindrical symmetry.

## 1. Introduction

In recent years the statics of charged dust distributions have attracted considerable attention. It has been shown that for such distributions the mass density is equal to the charge density (De and Raychaudhuri 1968). Bonnor (1965) has given a solution for static charged dust distributions with spherical symmetry which can be matched with the external Nordström solution. However, Som (1964, 1967) has shown that a static charged dust distribution with cylindrical symmetry cannot exist within a finite region nor in infinite space.

In this paper, a general diagonal line element is considered for the charged dust distribution and the metric elements are taken to be functions of one space coordinate. (The cylindrically symmetric case considered by Som is a special case of our line element.)

In § 2 two classes of solutions for the charged dust distribution are determined. In § 3 the conditions of continuity with the known matter-free space solutions (Raychaudhuri 1960) are considered. It is found that none of the solutions for the charged dust region can be matched with the matter-free space solutions. In § 4 the charged dust distributions are considered extending over all space and it is shown that even in that case no regular solution can exist.

## 2. Field equations and solutions for the charged dust region

Let us take the general diagonalized line element,

$$ds^2 = D\,dt^2 - A\,dx^2 - B\,dy^2 - C\,dz^2.$$

If we choose our coordinate system such that the static electric field is directed along one of our coordinate axes, say along the $x$ axis, then in the simplest case $A$, $B$, $C$ and $D$ become functions of $x$ alone.

The Einstein equations when written out explicitly are:

$$R_0^0 = F^{10}F_{10} - 4\pi\rho = \frac{1}{A}\left\{\frac{1}{4}\left(\frac{D'}{D}\right)^2 - \frac{D'}{4D}\left(\frac{A'}{A}+\frac{B'}{B}+\frac{C'}{C}\right) - \frac{D''}{2D} + \frac{D'A'}{2DA}\right\} \tag{2.1}$$

$$R_1^1 = F^{10}F_{10} + 4\pi\rho$$

$$= \frac{A'}{4A^2}\left(\frac{D'}{D}+\frac{B'}{B}+\frac{C'}{C}\right) + \frac{1}{2A}\left\{\frac{1}{2}\left(\frac{D'}{D}\right)^2 - \frac{D''}{D} + \frac{1}{2}\left(\frac{B'}{B}\right)^2 - \frac{B''}{B} + \frac{1}{2}\left(\frac{C'}{C}\right)^2 - \frac{C''}{C}\right\} \tag{2.2}$$

$$R_2^2 = -F^{10}F_{10} + 4\pi\rho = \frac{1}{4A}\left(\frac{B'}{B}\right)^2 - \frac{1}{4A}\left(\frac{B'}{B}\right)\left(\frac{D'}{D} - \frac{A'}{A} + \frac{C'}{C}\right) - \frac{B''}{2BA} \tag{2.3}$$

$$R_3^3 = -F^{10}F_{10} + 4\pi\rho = \frac{1}{4A}\left(\frac{C'}{C}\right)^2 - \frac{1}{4A}\left(\frac{C'}{C}\right)\left(\frac{D'}{D} - \frac{A'}{A} + \frac{B'}{B}\right) - \frac{C''}{2CA}, \tag{2.4}$$

where 0, 1, 2, 3 are the directions along $t$, $x$, $y$ and $z$ axes respectively. The primes represent differentiation with respect to $x$.

From the divergence of Einstein's equation,

$$|F^{10}| = \left|\frac{D'}{2D^{3/2}A}\right| \tag{2.5}$$

if we use the known result that in the case of static charged dust,

$$\sigma/\rho = \pm 1$$

that is,

$$-F_{10}F^{10} = \frac{1}{4A}\left(\frac{D'}{D}\right)^2. \tag{2.6}$$

Equations (2.1) to (2.4) may be replaced by the equivalent set,

$$R_2^2 = R_3^3 \tag{A}$$

$$R_0^0 + R_3^3 = 0 \tag{B}$$

$$R_0^0 + R_1^1 = 2F^{10}F_{10} \tag{C}$$

$$R_1^1 + R_2^2 = 8\pi\rho. \tag{D}$$

Let us write,

$$\left.\begin{aligned} \ln(BC) &= \beta \\ \ln(B/C) &= \alpha \\ \text{and} \quad \ln D &= \gamma \end{aligned}\right\} \tag{2.7}$$

then from (A) one can write,

$$\alpha'' + \frac{\alpha'\beta'}{2} + \frac{\alpha'}{2}\left(\frac{D'}{D} - \frac{A'}{A}\right) = 0. \tag{2.8}$$

By coordinate transformation, one can make $A = D^{-1}$, then (2.8) becomes,

$$\alpha'' + \frac{\alpha'\beta'}{2} + \alpha'\gamma' = 0. \tag{2.9}$$

From (B),

$$\frac{(\beta'' - \alpha'')}{2} + \frac{\beta'}{4}(\beta' - \alpha') + \beta'\gamma' - \frac{\alpha'\gamma'}{2} + \gamma'' + (\gamma')^2 = 0.$$

The above relation when combined with (2.9) reduces to,

$$\frac{\beta''}{2} + \frac{(\beta')^2}{4} + \beta'\gamma' + \gamma'' + (\gamma')^2 = 0. \tag{2.10}$$

From (C), together with the relation (2.6), one gets

$$\frac{\beta''}{2} + \frac{(\alpha')^2 + (\beta')^2}{8} + \frac{\beta'\gamma'}{2} + \gamma'' + \frac{(\gamma')^2}{2} = 0. \tag{2.11}$$

Of the Maxwell's equations,

$$F^{\mu\nu}{}_{;\nu} = 4\pi\sigma v^{\mu} \tag{2.12a}$$

$$*F^{\mu\nu}{}_{;\nu} = 0 \tag{2.12b}$$

(2.12*b*) is identically satisfied and from (2.12*a*) the charge density $\sigma$ can be determined.
From (D) and the relation (2.12*a*), one gets after simplification,

$$4\pi\rho = 4\pi|\sigma| = \frac{e^{\gamma}}{2}\left(\gamma'' + \frac{(\gamma')^2}{2} + \frac{\gamma'\beta'}{2}\right). \tag{2.13}$$

Obviously equations (2.9), (2.10), (2.11) are to be solved and (2.13) will determine $\rho$ which must be positive everywhere. Equation (2.9) shows that two cases may arise,

case I

$$\alpha' = 0, \tag{2.14a}$$

case II

$$\frac{\alpha''}{\alpha'} = -\left(\frac{\beta'}{2} + \gamma'\right). \tag{2.14b}$$

Case I. If equation (2.11) is subtracted from equation (2.10), one gets

$$\beta' + 2\gamma' = 0. \tag{2.15}$$

With (2.15), equations (2.10) and (2.11) are identically satisfied. Hence the metric components will have a degree of freedom corresponding to the arbitrariness in density distribution.

In view of equations (2.7), (2.14*a*), (2.15) and (2.13), the solution can be presented as

$$ds^2 = e^{\gamma}\, dt^2 - e^{-\gamma}(dx^2 + dy^2 + dz^2) \tag{2.16a}$$

and

$$8\pi\rho = 8\pi|\sigma| = e^{\gamma}\left(\gamma'' - \frac{(\gamma')^2}{2}\right) = e^{3\gamma/2}\frac{d}{dx}(\gamma' e^{-\gamma/2}). \tag{2.16b}$$

This shows that $\gamma' e^{-\gamma/2}$ must be an increasing function of $x$.

Case II. In this case, from (2.10) and (2.11) one gets,

$$(\alpha'+\beta'+2\gamma')(\alpha'-\beta'-2\gamma') = 0$$

that is, either (*a*) $\alpha'+\beta'+2\gamma' = 0$ or (*b*) $\alpha'-\beta'-2\gamma' = 0$.

Case II(*a*)

$$\alpha' \neq 0 \qquad \text{and} \qquad \alpha'+\beta'+2\gamma' = 0.$$

Then from (2.14*b*),

$$\frac{\alpha''}{\alpha'} = \frac{\alpha'}{2},$$

that is,

$$e^{\alpha} = (k_2-k_1^2x)^2 \tag{2.17}$$

where $k_1$ and $k_2$ are constants of integration. Equations (2.10) and (2.11) are identically satisfied with (*a*). After integration the condition (*a*) can be written as

$$\beta = -2\gamma-\alpha+k_3 \tag{2.18}$$

where $k_3$ is a constant of integration. In view of equations (2.7), (2.17) and (2.18), one can write after suitable transformation of coordinates,

$$A = e^{-\gamma}, \qquad B = e^{-\gamma}, \qquad C = x^2\, e^{-\gamma} \qquad D = e^{\gamma}.$$

Then the solution can be presented as,

$$ds^2 = e^{\gamma}\, dt^2-(dx^2+dy^2+x^2\, dz^2)\, e^{-\gamma} \tag{2.19a}$$

and

$$8\pi\rho = 8\pi|\sigma| = e^{\gamma}\left(\gamma''-\frac{(\gamma')^2}{2}+\frac{\gamma'}{x}\right) = \frac{e^{3\gamma/2}}{x}\frac{d}{dx}(x\gamma'\, e^{-\gamma/2}). \tag{2.19b}$$

Since $\rho$ is positive everywhere, $x\gamma'\, e^{-\gamma/2}$ must be an increasing function of $x$.

Case II(*b*)

$$\alpha' \neq 0 \qquad \text{and} \qquad \alpha' = \beta'+2\gamma'.$$

This is not a new case, however, and it only represents the possibility of interchange between $y$ and $z$ coordinates.

## 3. Matching with the matter-free space solution

If one starts with the same line element and with the assumption that only a source-free electric field remains along the $x$ direction, then the two sets of solutions come out (cf Raychaudhuri 1960).

When $\alpha' = 0$,

$$ds^2 = \left(\frac{a}{x^2}+\frac{b}{x}\right) dt^2-\left(\frac{a}{x^2}+\frac{b}{x}\right)^{-1} dx^2-x^2(dy^2+dz^2) \tag{3.1}$$

where $a$ and $b$ are constants, and when $\alpha' \neq 0$,

$$ds^2 = (C_1x^{\mu}+C_2x^{-\mu})^{-2}\, dt^2-(C_1x^{\mu}+C_2x^{-\mu})^2(dx^2+x^2\, dy^2+x^{(2-\lambda)}\, dz^2) \tag{3.2}$$

where $C_1$, $C_2$, $\lambda_1$ and $\mu$ are constants and $\lambda$ and $\mu$ are related by,

$$\mu^2 = \frac{\lambda}{4}(2-\lambda) \qquad \text{and} \qquad \mu \neq 0.$$

In the case $\lambda = 0$ or 2, that is $\mu = 0$, the line element becomes,

$$ds^2 = \left\{\ln\left(\frac{x+b}{a}\right)\right\}^{-2} dt^2 - \left\{\ln\left(\frac{x+b}{a}\right)\right\}^2 (dx^2 + dy^2 + x^2\, dz^2). \tag{3.3}$$

When matching of a charged dust distribution with the external solution is being considered across two parallel $yz$ planes, O'Brein and Synge boundary conditions are to be satisfied, that is $g_{\alpha\beta}$, $g'_{\alpha\beta}$ and $F^{10}F_{10}$ must be continuous across the boundary.

Case I. When $\alpha' = 0$: equation (3.1) can be transformed to a form in which the three-space metric is conformally euclidean, as

$$ds^2 = \frac{x^2}{(a/b - \frac{1}{4}bx^2)^2}\, dt^2 - \left(\frac{a}{b} - \frac{b}{4}x^2\right)^2 (dx^2 + dy^2 + dz^2). \tag{3.4}$$

If the external metric (3.4) is to be matched with the internal metric (2.16$a$) for the charged dust region, then from the continuity of $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ one finds that $m^2 = 1$ and $a/b = \frac{1}{4}b$ when the boundary is considered at $x = m$.

With these values for the constants, it is obvious that singularities appear across the boundary for the metric elements as well as for the electric field.

Case II. When $\alpha' \neq 0$: if the matching is considered between the external metric (3.2) and the metric for the charged dust region (2.19$a$) at a boundary $x = m$, from the continuity of $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ one finds that $m^\lambda = 1$ and $\lambda = 0$.

Obviously, then (3.2) cannot be matched with the inside solution.

Case II($a$). When $\alpha' \neq 0$ and $\lambda = 0$ or 2: in this case, when the external metric (3.3) is considered for matching with the internal metric (2.19$a$), $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ remain continuous. However, in order to make $F^{10}F_{10}$ continuous across the boundary, one must put $b = 0$ in (3.3). Hence in order to satisfy the O'Brein and Synge continuity conditions, the external metric should be,

$$ds^2 = \{\ln(x/a)\}^{-2}\, dt^2 - \{\ln(x/a)\}^2 (dx^2 + dy^2 + x^2\, dz^2). \tag{3.5}$$

However, from (2.19$b$), if the matter density is to be positive everywhere, $xy'\, e^{-\gamma/2}$ must be an increasing function of $x$. With the external metric (3.5), $xy'\, e^{-\gamma/2}$ is found to be $-2$ at both the boundaries, say $x = m$ and $x = n$. Hence $xy'\, e^{-\gamma/2}$ cannot be an increasing function of $x$ and the charged dust region cannot be matched with the matter-free space for any physically permissible situation.

## 4. Nonexistence of any regular solution

Case I. When $\alpha' = 0$: in order to make the density positive everywhere, $\gamma'\, e^{-\gamma/2}$ or $|d\, e^{-\gamma/2}/dx|$ must increase with $x$. If the absolute value of the rate of increase of $e^{-\gamma/2}$ continuously increases with $x$, then $|e^{-\gamma/2}| \to \infty$ as $x \to \infty$. Hence a singularity cannot be avoided in the metric elements.

Case II. When $\alpha' \neq 0$: then the matter density is given by (2.19*b*). However if one puts $e^{-\gamma/2} = y$, equation (2.19*b*) can also be written as,

$$\frac{d}{dx}(xy') = -4\pi\rho x y^3. \tag{4.1}$$

A little argument shows that in (4.1) if $\rho$ is to be positive everywhere, $y \to -\infty$ as $x \to \infty$. This contradicts the condition that $y$ must be positive everywhere. Hence equation (2.19*b*) does not admit any regular solution with positive matter density everywhere.

Hence, for a general diagonalized line element with the metric elements being functions of only one space coordinate, no regular solution can be found for the charged dust distributions.

This is an extension of Som's result for the case of cylindrical symmetry.

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